Antiphase synchronization in coupled chaotic oscillators

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Anti-phase synchronization (AS) in coupled chaotic oscillators is investigated. The necessary condition for AS is given and the stability of AS is studied. Results are demonstrated with numerical simulations and electronic circuits.

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The study of synchronization phenomena in periodic dynamical systems has been active since the earlier days of physics. Recently, the investigation on this phenomenon has been extended to coupled chaotic oscillators. Various types of chaos synchrony such as complete synchronization (CS) [1,2], generalized synchronization (GS) [3,4], phase synchronization (PS) [5], and lag synchronization (LS) [6] have been described. CS is the simplest form of synchronization where the distance between the states of interacting identical systems approaches zero for $t\to\infty$. GS implies the hooking of the output of one system to a given function of the output of the other system. PS is characterized by the phase differences among chaotic oscillators which are locked within 2π. LS is an intermediate state between PS and CS, which is described as the coincidence of two chaotic trajectories with a constant time lag. There is another type of synchrony, anti-phase synchronization, in coupled chaotic systems which does not receive enough attentions.

It is well known that anti-phase synchronization (AS) is the first observation on synchrony of two oscillators by Huygens four hundreds years ago. Huygens found that the pendulum clocks which are suspended by the side of each other swung in exactly the same frequency and π out of phase. Huygens's observations have been reconsidered by the group in Georgia Institute of Technology experimentally and theoretically [7]. The authors pointed out that AS is dominant for weak coupling between identical plane pendulums hang from a common rigid frame. The pendulums in these studies are phase oscillators with constant amplitude. However for chaotic oscillators, the amplitude of oscillation varies with time. Therefore, to study AS in coupled chaotic oscillators, we have to take the amplitude besides phase of the oscillator into consideration. Otherwise, it will fall into the category of PS. As an extension of AS in coupled phase oscillators, we define AS in chaotic oscillators as the phenomenon where the variables of two interacting oscillators have the same amplitude but differ in sign. So far, the term of AS in coupled chaotic systems has been used in literatures for different purposes [8–12]. One usage of AS can be investigated in the scope of PS where two oscillators has a phase difference of π but differ in amplitude [8]. Another case studied by Cao and Lai is in the frame of master-slave system [9]. AS occurs when the slave and its replica have the same amplitude but differ in the signs with respect to each other. This case is more likely to be related to GS.

The main goal in this work is to investigate AS occurred in a system of two linearly coupled identical chaotic oscillators. In particular, we give the necessary conditions for the occurrence of AS and discuss the criterions for the stability of AS. The statements are demonstrated by numerical simulations and electronic circuits. We also show that an AS state is possible to coexist with the CS state. Furthermore, we show that AS can occur in a ring of oscillators.

The model we consider takes the general form

$$\dot{\mathbf{x}}_1=\mathbf{f}(\mathbf{x}_1)+\epsilon\mathcal{D}(\mathbf{x}_2-\mathbf{x}_1),$$

$$\dot{\mathbf{x}}_2=\mathbf{f}(\mathbf{x}_2)+\epsilon\mathcal{D}(\mathbf{x}_1-\mathbf{x}_2), \quad (1)$$

where $\mathbf{x}_i\in R^n(i=1,2)$, $\mathbf{f}:R^n\to R^n$ is nonlinear and capable of exhibiting rich dynamics such as chaos, ϵ is a scalar coupling constant, and $\mathcal{D}$ is a constant matrix describing coupling scheme. We say that Eq. (1) possesses the property of AS between $\mathbf{x}_1$ and $\mathbf{x}_2$ if there exists an anti-phase synchronous manifold (ASM) $M=\{(\mathbf{x}_1,\mathbf{x}_2):\mathbf{x}_1=-\mathbf{x}_2=\mathbf{x}^*(t)\}$ such that trajectories of Eq. (1) with certain initial conditions approach M as time goes to infinity. The motion on ASM is described by

$$\dot{\mathbf{x}}^*=\mathbf{f}(\mathbf{x}^*)-2\epsilon\mathcal{D}\mathbf{x}^*,$$

$$-\dot{\mathbf{x}}^*=\mathbf{f}(-\mathbf{x}^*)+2\epsilon\mathcal{D}\mathbf{x}^*, \quad (2)$$

To keep the compatibility between two equations in Eq. (2), we have the necessary condition for AS: *the nonlinear function* $\mathbf{f}(\mathbf{x})$ *is an odd function of* $\mathbf{x}$, *that is*, $\mathbf{f}(\mathbf{x})=-\mathbf{f}(-\mathbf{x})$. It is important to note that the anti-phase synchronous state is not the solution of the isolated oscillator any more. The stability of the AS state can be determined by letting $\mathbf{x}_i=\mathbf{x}^*+\xi_i$, $(i=1,2)$ and linearizing Eq. (1) about $\mathbf{x}^*(t)$. This leads to

$$\frac{d}{dt}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=\begin{pmatrix}D\mathbf{f}(\mathbf{x}^*) & 0\\ 0 & D\mathbf{f}(-\mathbf{x}^*)\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}+\epsilon\mathcal{D}A\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} \quad (3)$$

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where $D\mathbf{f}(\mathbf{x}^*)$ is the Jacobian of $\mathbf{f}$ on $\mathbf{x}^*$, and $A=\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}$. Since $\mathbf{f}(\mathbf{x})$ is an odd function of $\mathbf{x}$, we have $D\mathbf{f}(\mathbf{x}^*)=D\mathbf{f}(-\mathbf{x}^*)$ and linear stability equations can be diagonalized by expanding into the eigenvectors of A, $\xi=\Sigma_{i=1}^{2}\eta_i\phi_i$. Carrying this out gives

$$\dot{\eta}_i = [D\mathbf{f}(\mathbf{x}^*) + \epsilon\lambda_i\mathcal{D}]\eta_i,$$

$$i = 0,1 \tag{4}$$

where $\lambda_i=0,-2$ are the eigenvalues of A. It can be demonstrated that ASM coincides with the subspace spanned by the eigenvector of A with eigenvalue $\lambda=-2$. The $\lambda=0$ mode governs the motion transversal to ASM and this mode has Lyapunov exponents $\Lambda_1^{(0)}\geqslant\Lambda_2^{(0)}\geqslant\cdots\geqslant\Lambda_n^{(0)}$. Therefore, *AS is stable if and only if* $\Lambda_1^{(0)}<0$. The dynamics on ASM, periodicity, or chaoticity, can be known from the $\lambda=-2$ mode and it is possible to observe rich dynamics for the state of AS no matter how the isolated oscillator behaves. It has to be stressed that the stability problem of AS is different from that for CS since the evolution equations underlying the stability analysis for them are different: Equation (2) for AS while $\dot{\mathbf{x}}^*=\mathbf{f}(\mathbf{x}^*)$ for CS.

The descriptions above are made on complete AS. However it is possible to find partial AS where only part of variables is in AS while the rest in CS. Considering an n-dimensional system, $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$, whose state variables can be decomposed into two groups $\mathbf{x}=\mathbf{y}\oplus\mathbf{z}$ where AS vector $\mathbf{y}\in R^m$ and CS vector $\mathbf{z}\in R^l$ with $m+l=n$. The motion equations are described as: $\dot{\mathbf{y}}=\mathbf{g}(\mathbf{y},\mathbf{z})$ and $\dot{\mathbf{z}}=\mathbf{h}(\mathbf{y},\mathbf{z})$. Following the line as Eq. (2), we have the necessary conditions for partial AS: $\mathbf{g}(\mathbf{y},\mathbf{z})=-\mathbf{g}(-\mathbf{y},\mathbf{z})$ and $\mathbf{h}(\mathbf{y},\mathbf{z})=\mathbf{h}(-\mathbf{y},\mathbf{z})$. The stability analysis for partial AS is a little difficult. Since $D\mathbf{f}(\mathbf{x}^*)\neq D\mathbf{f}(-\mathbf{x}^*)$, here the linearization equation (Eq. (3)) cannot be diagonalized by expanding into the eigenvectors of A and we cannot get a concise form like Eq. (4). Nevertheless, we may investigate the stability of partial AS by using new variables $\mathbf{u}=(\mathbf{x}_1+\mathbf{x}_2)/2$ and $\mathbf{v}=(\mathbf{x}_1-\mathbf{x}_2)/2$. The stability of AS depends on whether $\mathbf{u}$ goes to zero as $t\to\infty$. In some systems displaying partial AS, the linearization equation for $\mathbf{u}$ corresponds to the mode $\lambda=0$ and for $\mathbf{v}$ to mode $\lambda=-2$ if nonlinear term is quadratic.

To illustrate the phenomenon of AS, let us go to specific systems. The first example is two identical Chua circuits linearly coupled together. The system is defined as

$$\dot{x}_i = \alpha(y_i - \phi(x_i)),$$

$$\dot{y}_i = x_i - y_i + z_i,$$

$$\dot{z}_i = -\beta y_i + \epsilon(y_j - y_i), \quad (i,j = 1,2) \tag{5}$$

where $\phi(x)=bx+(b-a)$ for $x<-1$, $\phi(x)=ax$ for $-1<x<1$, and $\phi(x)=bx+(a-b)$ for $x>1$. Equation (5) has a chaotic attractor for parameters $\alpha=9$, $\beta=14.6$, $a=-1/7$, and $b=2/7$. When β increases, the isolated system undergoes an inverse period-doubling bifurcation to period-1 solution. The right hand sides of Eqs. (5) are odd functions of the variables x,y,z and AS is possible based on the analysis above. The time-series in Fig. 1(a) show a state where two circuits os-

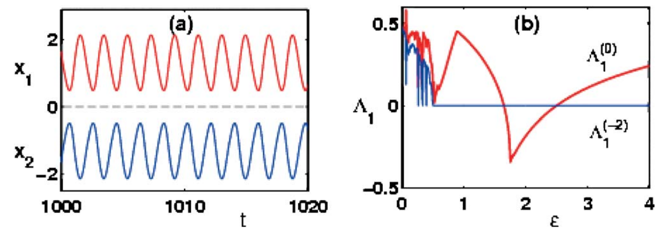

FIG. 1. (Color online) (a) The time series of variables $x_i, i=1,2$ for the coupled identical Chua circuits. $\epsilon=2$. The dashed line is for the sum x_1+x_2 which stays at zero. The state of AS here is periodic. (b) The largest conditional Lyapunov exponents for the $\lambda=0$ and $\lambda=-2$ modes against ϵ are plotted.

cillate in an anti-phase way when the coupling constant $\epsilon=2$. The state of AS can be proved further by the sum of x_1+x_2 which stays at zero all the time. Clearly, each circuit falls onto a periodic orbit which is not the solution of the isolated Chua circuit. The largest Lyapunov exponents against coupling constant ϵ for the $\lambda=0$ and $\lambda=-2$ modes are plotted in Fig. 1(b). $\Lambda_1^{(0)}$ is negative in the regime of $1.6<\epsilon<2.5$, which indicates the stable AS state. $\Lambda_1^{(-2)}$ stays at zero for $\epsilon>0.6$ and the resulting stable AS state is periodic. The periodic state of AS is dominant in Chua circuits, but it is possible to find chaotic state of AS in other systems such as Saito's oscillator [13] (results will be presented elsewhere).

In the second example, the isolated element is the Lorenz system

$$\dot{x} = \sigma(y - x),$$

$$\dot{y} = rx - y + xz,$$

$$\dot{z} = xy - \beta z, \tag{6}$$

Where $\sigma=10$, $r=28$, and $\beta=8/3$. Since the vector field of the Lorenz system satisfies $\mathbf{f}(x,y,z)=-\mathbf{f}(-x,-y,z)$, it is possible to observe partial AS. CS in coupled Lorenz systems has been discussed for different coupling schemes [14]. Partial AS was only discussed recently [10,11] where $x_1(t)=-x_2(t)$, $y_1(t)=-y_2(t)$, and $z_1(t)=z_2(t)$ occurs. However, we have to point out that the partial AS in Ref. [10], where $\mathcal{D}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$ is not greatly interesting. Since the Lorenz system is invariant under the transformation $x\to-x$, $y\to-y$, and $z\to z$, $(-x(t),-y(t),z(t))$ is also the Lorenz systems's solution if $[x(t),y(t),z(t)]$ satisfies Eq. (6). For $\mathcal{D}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$, the solution of the partial AS can be obtained from that of the CS by changing the sign of variables x and y of one element. Furthermore it can be shown that the linear stability of the AS state in this case does not involve the property of the AS state and is the same as the corresponding CS state. Therefore this type of partial AS is stable if and only if CS is

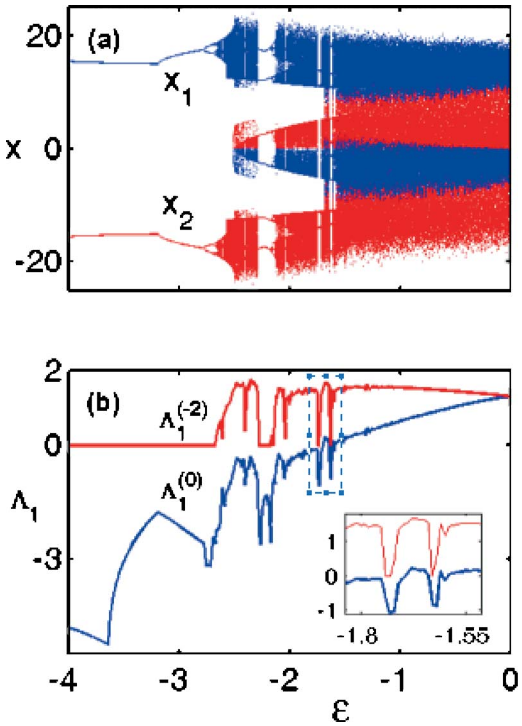

FIG. 2. (Color online) (a) The bifurcation diagrams for two oscillators against ϵ are presented. In the state of AS, two bifurcation diagrams are symmetrical about the horizontal axis. (b) The largest conditional Lyapunov exponents for the $\lambda=0$ and $\lambda=-2$ modes against ϵ are plotted.

stable. If the coupling scheme D involves the variable x or y, we may have another type of partial AS, which cannot be constructed from CS state, whose stability does not depend on CS. We let $\mathcal{D}=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$. It is indicative to plot the bifurcation diagram of coupled Lorenz systems against coupling constant ϵ. The results are displayed in Fig. 2(a) where the values of x_1 and x_2 for each ϵ are recorded at the same time when x_1 reaches its maximum values. The bifurcation diagrams show that the size of the attractor for the coupled system undergoes a sharp change at around $\epsilon_c=-1.7$ [15]. Below ϵ_c, the bifurcation diagrams for two elements are symmetrical about horizontal axis and the coupled system undergoes transitions from chaos to period-1 solution by inverse cascade of period-doubling bifurcations as ϵ decreases. The largest Lyapunov exponents against the coupling constant ϵ for the $\lambda=0$ and $\lambda=-2$ modes are plotted in Fig. 2(b). It is clear that $\Lambda_1^{(0)}$ for the transversal mode crosses zero at the parameter where the sharp change on the size of attractor happens, which means that AS is stable for $\epsilon<-1.7$. It is also interesting to find rich dynamics for the AS state such as chaos, periodic behaviors, periodic windows and so on.

The experimental test of AS can be carried out since both Chua circuit and Lorenz system allow for easy circuit implementation. We build the coupled circuits for the Chua and Lorenz systems as Refs. [16,17]. The results are presented in Fig. 3 where we observe the periodic AS state in the Chua circuits [Fig. 3(a)] and chaotic AS state in the Lorenz circuits [Fig. 3(b)].

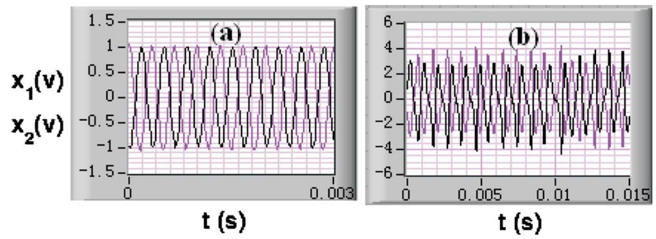

FIG. 3. (Color online) The demonstration of the phenomenon of AS in electronic circuits. (a) Chua circuit; (b) The Lorenz system. The coupling schemes are the same as those in numerical simulations.

For practical applications, it is worthy of investigating the coexistence of CS and AS. Considering the fact that the stability equation for the $\lambda=-2$ mode is the linearization of Eq. (2) (the evolution equation for the motion on ASM), Eq. (4) can be reformulated as

$$\dot{\eta}_i=[D\mathbf{f}'(\mathbf{x}^*)+\epsilon\lambda_i'\mathcal{D}]\eta_i,\quad i=0,1 \tag{7}$$

where $\mathbf{f}'(\mathbf{x})=\mathbf{f}(\mathbf{x})-2\epsilon\mathcal{D}\mathbf{x}$ and $\lambda_i'=\lambda_i+2$. Then we recover the similar description to the stability problem of CS: The $\lambda'=0$ mode is in charge of the motion on ASM while the transversal motion is represented by the $\lambda'\neq 0$ mode. The difference between CS and AS about the motion on synchronous manifold is not crucial on the problem of coexistence. The main difference is that the transversal mode for CS is characterized by negative λ while the transversal mode for AS by positive λ'. Therefore, if the stability equation Eq. (7) could give negative $\Lambda_1^{(0)}$ for both positive and negative $\epsilon\lambda'$, it is possible to observe the coexistence of CS and AS. For the examples discussed above, the stability equation only gives negative Λ_1^0 for negative $\epsilon\lambda'$ in Lorenz system and positive $\epsilon\lambda'$ in Chua circuit. Therefore, in coupled Lorenz systems, CS occurs for positive ϵ while AS for negative ϵ. In the contrary, the coupled Chua circuits display AS for positive ϵ. The coexistence of CS and AS can be found in the system [18]

$$\dot{x}_i=y_i-\beta z_i,$$

$$\dot{y}_i=-x_i-2\gamma y_i+\alpha z_i+\epsilon(y_j-y_i),$$

$$\dot{z}=(x_i-z_i^3+z_i)/\mu,\quad (i,j=1,2) \tag{8}$$

where $\beta=1$, $\gamma=0.26$, $\alpha=0.165$, and $\mu=0.4$. The isolated oscillator displays chaotic behavior. When the coupling constant $\epsilon\in(0.34,0.52)$, AS state coexists with CS one. The basins of attraction for AS and CS attractors are shown in Fig. 4 for $\epsilon=0.5$ in the x_1-x_2 plane.

To complete the investigation, the phenomenon of AS is studied on a ring with $2N$ identical oscillators. A little different from the case of two coupled oscillators, the state of AS here satisfies $\dot{\mathbf{x}}^*=\mathbf{f}(\mathbf{x}^*)-4\epsilon\mathcal{D}\mathbf{x}^*$. Correspondingly, the stable regime of ϵ is shrunk. As a result, it is hard to observe AS in a ring of Chua circuits according to the knowledge from Fig. 1. However, the state of AS in a ring of the Lorenz systems

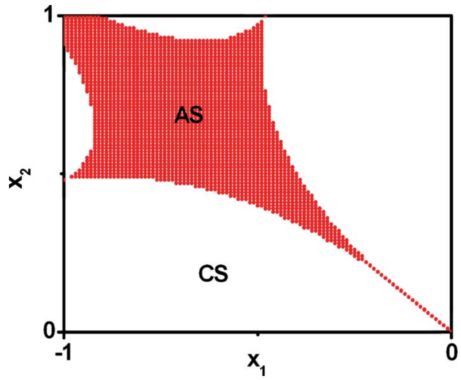

FIG. 4. (Color online) The basins of attraction for AS and CS attractors in the $x_1 - x_2$ plane with initial conditions $x_1 = y_1 = z_1$ and $x_2 = y_2 = z_2$ for $\epsilon = 0.5$

is relatively easy to be realized. For example, the state of AS for 10 coupled identical Lorenz oscillators is shown Fig. 5.

In discussion, AS in coupled identical oscillators is a prevailing phenomenon that happens for isolated oscillators with symmetry. Either complete AS or partial AS is not limited to the special coupling schemes. Different linear coupling schemes usually lead to different stable parameter regions for AS and different bifurcation sequences for AS. AS could be observed in laser systems experimentally since the possible equivalence between laser system and Lorenz oscillator [19]. Finally, AS has potential application in practice fields such as secure communications.

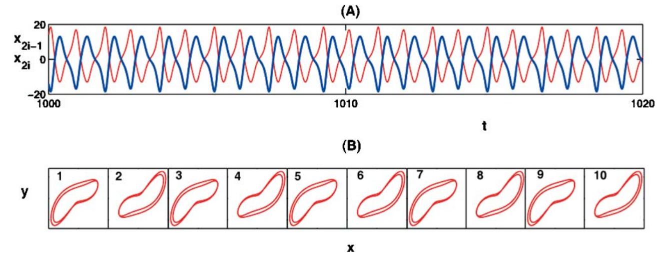

FIG. 5. (Color online) AS in a ring with 10 identical Lorenz oscillators. $\epsilon = 1.35$. (a) The time series of 10 oscillators are plotted where the even nodes are in anti-phase with the odd ones. (b) The phase trajectories for the 10 oscillators are projected onto the plane $x - y$.

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